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Tensor Product of Frame Manuals

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Since its first use, there has been widespread dissatisfaction with the Hilbert-space tensor product as a device for coupling the Hilbert-space models of two separated quantum mechanical systems. The Hilbert-space model is paraphrased manual-theoretically by the assertion that quantum mechanical entities are represented by frame manuals. There is a natural, heuristically straightforward tensor product for (unital) manuals, and it is natural therefore to ask whether the tensor product of frame manuals might serve as an alternative model of separated quantum mechanical systems. It is shown that the states on a tensor product of complex frame manuals give rise uniquely to sesquilinear forms on the tensor product of the underlying Hilbert spaces. In certain cases, these in turn give rise to operators, which, however, are not generally positive, and which, even if compact, need not be trace-class.

1. INTRODUCTION

Orthodox nonrelativistic quantum mechanics represents physical systems (e.g., particles) in terms of separable complex Hilbert spaces: The unit vectors of such a Hilbert space play a dual role, representing both the pure states of such a system and the outcomes of the "maximally informative" discrete experiments that can be performed on the system. This is recognized in Dirac's formalism, where the unit vector ψ is denoted by $|\psi\rangle$ when it plays the role of a state, and by $\langle\psi|$ when it plays the role of an outcome.

It is fundamental to QM that (pure) states may be superposed. It is also a feature of QM that every pure state corresponds to a unique outcome which is certain to occur in that state, and vice versa. Hence, QM allows implicitly for the superposition of *outcomes.*

Let us take it for granted that QM is a reasonable description of "primitive" or "elementary" systems, without accepting that it correctly

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describes all systems. Then we ask how we are to model a system consisting of two *noninteracting* subsystems—for example, a pair of spacelike-separated particles. Traditionally, if the two subsystems are represented by Hilbert spaces H and K, respectively, then the composite is taken to be represented by the tensor product $H \otimes K$. This is justified by an appeal to the superposition principle (and to the assumption that QM again describes the composite): The larger system should admit states of the form (ϕ, ψ), where ψ is a state of the first system and ψ is a state of the second system. Forming superpositions and taking the closure of the resulting pre-Hilbert space (with the obvious inner product) yields the desired tensor product of $H \otimes K$.

This is plausible enough as far as states are concerned, but (unless we accept the universality of QM), is problematic concerning the superposition of outcomes: There is no physically plausible sense in which the outcomes of experiments can be "superposed." Surely, given an outcome ϕ of an experiment performable on the first system and an outcome ψ of an experiment performable on the second, $\phi \otimes \psi$ represents an outcome of an experiment performable on the second system. However, if the two systems are separated, it is not clear that a nonpure tensor τ belonging to the unit sphere of $H \otimes K$ represents an outcome of any experiment on the joint system. In particular, the assumption that a state on the joint system should assign nonnegative probabilities to such pure tensors is not justified, nor is it even justified *a priori* to assume that a consistent assignment of probabilities to pure tensors gives rise to an operator on $H \otimes K$.

In what follows, we will discuss the consequences of relaxing the assumption that the composite of noninteracting quantum entities is described by quantum mechanics. We will argue that a reasonable model of such a system leads to a description of states as sesquilinear forms on $H \otimes K$ which are *positive on pure tensors* and have unit trace on frames consisting of pure tensors. We will recover a representation of such forms as operators, not on $H \otimes K$, but on $H \otimes \bar{K}$, where $\bar{K}$ is the Hilbert space conjugate to K. This work extends to infinite-dimensional Hilbert spaces the results of Kläy *et al.* (1987).

2. FRAME MANUALS

The axiomatics of QM can be reduced to a single statement: The sample spaces of the experiments needed to describe a quantum mechanical entity may be identified with the frames (i.e., orthonormal bases) of a separable complex Hilbert space H. If one defines a *state* of such an entity to be a simultaneous assignment of a probability weight to each of these sample spaces—thus, a nonnegative function on H's unit sphere that sums to 1 along every frame—then Gleason's theorem asserts that (as long as dim $H > 2$), every such state ω is uniquely representable by a positive normalized trace-class operator T_ω such that for any outcome (unit vector) ψ, $\omega(\psi) = \langle T_\omega \psi, \psi \rangle$. The theorems of Stone, Wigner, and Mackey allow one to recover the traditional dynamics of "elementary" particles within this framework.

In order to speak clearly about the probabilistic notions implicit in this discussion, it is helpful to abstract as follows (Foulis and Randall, 1981; Gudder, 1988): An entity (of whatever kind) is described in terms of the admissible experiments whereby its properties can be investigated. It is reasonable to assume that these experiments are discrete, and to identify such an experiment with its sample space. One may wish to identify outcomes of distinct experiments (where one believes that these experiments "mean the same thing"). One is left with a collection $\mathscr{A}$ of possibly overlapping sets, each understood as the sample space of an experiment. One naturally defines a *state* of such an entity to be a function $\omega: \bigcup(\mathscr{A}) \to [0, 1]$ such that $\sum_{x \in E} \omega(x) = 1$ for every $E \in \mathscr{A}$. Let $\Omega(\mathscr{A})$ denote the set of all such states. A subset A of an element E of $\mathscr{A}$ is naturally referred to as an *event*. Given a state ω and an event A, we write $\omega(A)$ for $\sum_{x \in A} \omega(x)$—that is, $\omega(A)$ is the probability that the event A will be confirmed, in state ω, if tested.

A collection of sets $\mathscr{A}$ thus interpreted (and subject to certain additional conditions that we will be able to ignore) is called a *manual*. This notion is illustrated by the following examples: (i) If (S, Σ) is a measurable space, let $\mathscr{A}$ denote the collection of countable partitions of S by Σ-measurable sets. Then $\Omega(\mathscr{A})$ consists exactly of the σ-additive probability measures on (S, Σ). (ii) Let H be a separable complex Hilbert space and let $\mathscr{F}(H)$ denote the set of all frames of H. This is the *frame manual* of H.

Consider a system consisting of two subsystems which retain enough individuality that one may perform experiments on each separately. Suppose that the subsystems are represented by manuals $\mathscr{A}$ and $\mathscr{B}$, respectively. Given an experiment $E \in \mathscr{A}$ and an experiment $F \in \mathscr{B}$, we allow that $E \times F$ describes a legitimate experiment on the composite system (e.g., the experiments E and F may be performed independently on the subsystems and the results collated at some later time). If the system represented by $\mathscr{A}$ does not exert any causal influence on the second system, so that the execution of an experiment $E \in \mathscr{A}$ does not affect the state of the system represented by $\mathscr{B}$, then we must also allow compound experiments, in which an initial experiment E belonging to $\mathscr{A}$ is executed, and, if the outcome x is secured, a definite experiment $F_x \in \mathscr{B}$ is subsequently performed.

It will be convenient to adopt a juxtapositive notation for ordered pairs of outcomes. Thus, we will write xy instead of (x, y), AB instead of $A \times B$, where A and B are events, and xA instead of $\{x\} \times A$, where x is an outcome and A an event. Then, clearly, the outcome set for a compound experiment such as described above is

$$\bigcup_{x\in E} xF_x$$

where $E\in\mathscr{A}$ and $F_x\in\mathscr{B}$ for each $x\in E$. Similarly, if the second system does not influence the first, compound experiments of the form

$$\bigcup_{y\in F} E_y y$$

where $F\in\mathscr{B}$ and for each outcome $y\in F$, $E_y\in\mathscr{A}$ must be allowed. The collection of all such experiments is denoted $\mathscr{A}\mathscr{B}$ [for more details, consult Foulis and Randall (1985) or Kläy (1988)].

Let us consider the states on $\mathscr{A}\mathscr{B}$: It is not difficult to see that $\bigcup\mathscr{A}\mathscr{B}=\bigcup\mathscr{A}\times\bigcup\mathscr{B}$. Since the product operations EF belong to $\mathscr{A}\mathscr{B}$, a state on $\mathscr{A}\mathscr{B}$ must sum to 1 over such operations. Denoting the collection of product operations by $\mathscr{A}\times\mathscr{B}$ (if the abuse may be forgiven), one sees that $\Omega(\mathscr{A}\mathscr{B})$ consists of those $\omega\in\Omega(\mathscr{A}\times\mathscr{B})$ having the property that $\omega(Ey)$ and $\omega(xF)$ are independent of E and F for every x and y, i.e., those states that reveal no influence of the one system on the other when one conditions by the choice of an experiment.

In discussing states on manuals, it is helpful to introduce the span of $\Omega(\mathscr{A})$ in $\mathbf{R}^X$, where $X=\bigcup(\mathscr{A})$. This span is called the *signed weight space* of $\mathscr{A}$, and denoted by $V(\mathscr{A})$. We refer to an element of the positive cone of $V(\mathscr{A})$ (ordered pointwise on X) as a *positive weight* on $\mathscr{A}$. The Ω is a base for this positive cone, and it can be shown (Cook, 1985) that $V(\mathscr{A})$ is always complete in the associated base-norm.

Kläy *et al.* (1987) were able to compute the dimension of $V(\mathscr{A}\mathscr{B})$ when this is finite and were thereby able to prove the following:

Theorem 2.1. If $V(\mathscr{A})$ and $V(\mathscr{B})$ are finite dimensional, then

$$V(\mathscr{A}\mathscr{B})\simeq V(\mathscr{A})\otimes V(\mathscr{B})$$

$V(\mathscr{A}\mathscr{B})$ is characterized for arbitrary manuals $\mathscr{A}$ and $\mathscr{B}$ as follows (Wilce, 1989).

Proposition 2.2. For any $\mathscr{A}$ and $\mathscr{B}$, $V(\mathscr{A}\mathscr{B})$ is isomorphic to the positively generated part of the space of weak*-to-weakly continuous linear operators $f\colon V^*(\mathscr{A})\to V(\mathscr{B})$.

This result (of which Theorem 2.1 is a special case) is a consequence of the following simple observation, which we will also exploit.

Lemma 2.3. The positive weights on $\mathscr{A}\mathscr{B}$ are in one-to-one correspondence with the set of nonnegative functions ω on XY such that (i) for every product operation EF in $\mathscr{A} \times \mathscr{B}$,

$$\sum_{xy \in EF} \omega(xy) < \infty$$

and (ii), for every $x \in X$, $y \in Y$, the functions $\omega(x\cdot)$ and $\omega(\cdot\, y)$ are positive weights on $\mathscr{A}$ and $\mathscr{B}$, respectively.

We now restate our problem: If we accept that each of a pair of noninteracting physical systems is represented by a frame manual—say, by $\mathscr{F}(H)$ and $\mathscr{F}(K)$, respectively—then the composite system is (at worst, under-) described by the manual $\mathscr{F} := \mathscr{F}(H)\mathscr{F}(K)$. What can be said about $\Omega(\mathscr{F})$?

3. STATES ON A TENSOR PRODUCT OF FRAME MANUALS

Let H, K, and $\mathscr{F}$ be as described above. There is a natural interpretation $\mathscr{F} \to \mathscr{F}(H \otimes K)$ given by $\psi\phi \mapsto \psi \otimes \phi$. Notice that this is not an injection, since for $|a| = 1$, $(a\phi)\psi = \phi(a\psi)$ in $\mathscr{F}$. Every density operator W on $H \otimes K$ determines a state ω on $\mathscr{F}$ by $\omega(\phi\psi) = \langle W\phi \otimes \psi, \phi \otimes \psi \rangle$.

Kläy *et al.* (1987) apply Theorem 2.1 to show that, if the Hilbert spaces H and K are finite dimensional (and have dimension >2), then, conversely, every state on $\mathscr{F}$ could be represented uniquely by a self-adjoint operator W on $H \otimes K$ which is *positive on pure tensors*:

$$\langle W\phi \otimes \psi, \phi \otimes \psi \rangle \geq 0$$

In this section, we show that, in general, states on $\mathscr{F}$ can be represented, if not by operators, at least by sesquilinear forms on the *algebraic* tensor product of the two Hilbert spaces. To avoid constant repetition, we adopt the convention that all Hilbert spaces under consideration have dimension >2.

Definition 3.1. Let $H \odot K$ denote the algebraic tensor product of H and K. We will say that a state $\omega \in \mathscr{F}$ is *represented* by a sesquilinear form Φ on $H \odot K$ iff for all unit vectors $\phi \in H$ and $\psi \in K$,

$$\omega(\phi\psi) = \Phi(\phi \otimes \psi, \phi \otimes \psi)$$

in which case we write $\omega \sim \Phi$. If A is an operator on $H \odot K$ and Φ is the sesquilinear form $\langle A\cdot, \cdot \rangle$, we will say that ω is represented by A, and write $\omega \sim A$.

Recall (Kadison and Ringrose, 1983, pp. 125–143) that a sesquilinear form Φ on $H \times K$ is *Hilbert-Schmidt* iff there exist frames E and F for H and K, respectively, such that

$$\sum_{x,y \in EF} |\Phi(x, y)|^2 < \infty$$

and that a sesquilinear form lifts to a continuous functional on the tensor product $H \otimes \bar{K}$ iff it is Hilbert-Schmidt. Also, recall that an operator A on H is HS iff the associated form $\langle A\cdot, \cdot\rangle$ is HS. Immediately we have the following result.

Lemma 3.2. A state $\omega \in \Omega(\mathscr{F})$ is representable by an operator iff $\omega \sim \Phi$ for some form Φ such that $\Phi(\phi \otimes \psi, \eta \otimes \zeta)$ is HS in (η, ζ) for each pure tensor $\phi \otimes \psi$, and is represented by a *bounded* operator iff Φ is Hilbert-Schmidt in (ϕ, ψ) and (η, ζ) separately.

Lemma 3.3. Let $\omega \sim \Phi$ and $\omega \sim \Psi$. Then $\Phi = \Psi$.

Proof. It suffices to show that a sesquilinear form Φ on $H \otimes K$ is uniquely determined by the biquadratic form $\Phi(\phi \otimes \psi, \phi \otimes \psi)$. The form $\Phi(\phi \otimes \psi, \eta \otimes \zeta)$, which clearly determines Φ, is sesquilinear in ψ and ζ, and hence, for fixed ϕ and η, is determined by the quadratic form $\Phi(\phi \otimes \psi, \eta \otimes \psi)$. Now, fixing ψ and polarizing again, the original form is seen to depend only on $\Phi(\phi \otimes \psi, \phi \otimes \psi)$. ■

Corollary 3.4. A PPT operator is necessarily self-adjoint.

Definition 3.5. A sesquilinear form Φ on $H \odot K$ is *positive on pure tensors* (PPT) iff $\Phi(\phi \otimes \psi, \phi \otimes \psi) \geq 0$ for all $\phi \in H$ and $\psi \in K$. A PPT form Φ has *finite trace* iff there exist frames E of H and F of K such that

$$\mathrm{Tr}(\Phi) := \sum_{\phi\psi \in EF} \Phi(\phi \otimes \psi, \phi \otimes \psi)$$

is finite. We will speak of an operator A on $H \otimes K$ as being PPT or having finite trace iff the associated sesquilinear form is PPT or has finite trace.

Lemma 3.6. Let Φ be PPT with finite trace. Then $\mathrm{Tr}(\Phi)$ is well defined.

Proof. Define ω on XY (the product of the unit spheres of H and K) by $\omega(\phi\psi) = \Phi(\phi \otimes \psi, \phi \otimes \psi)$. Then, for fixed ϕ, ω is a quadratic form in ψ summable over some frame of K. But then $\omega(\phi, \cdot)$ is the form determined by a trace-class operator, whence it sums to a common, finite value over all frames of K. Similarly, $\omega(\cdot, \psi)$ sums to a common value over all frames of H. Since ω sums to a finite value over a product frame EF, it sums to the same value over all product frames. ■

The preceding lemmas together with Lemma 2.1 add up to a proof of the following.

Proposition 3.7. Every state on $\mathscr{F}$ is represented by a unique sesquilinear form on $H \odot K$. A sesquilinear form on $H \odot K$ represents a state on $\mathscr{F}$ iff it is PPT with trace 1.

Since $\mathscr{F}$ and $\mathscr{F}(H)\times\mathscr{F}(K)$ have the same outcomes, we may speak of states on the latter as being represented by forms or by operators on $H\odot K$. However, we have the following result.

Corollary 3.8. A state on $\mathscr{F}(H)\times\mathscr{F}(K)$ is representable by a sesquilinear form on $H\odot K$ iff it is a state of $\mathscr{F}$.

Thus, the distinguishing feature of influence (as here described) is nonlinearity: A state on $\mathscr{F}(H)\times\mathscr{F}(K)$ reflects the influence of one system on the other iff it is not a state on $\mathscr{F}(H)\otimes\mathscr{F}(K)$, i.e., iff it does not extend to a sesquilinear form on $H\odot K$.

Notice that every positive operator is PPT, and that a positive operator has finite trace iff it is trace-class. The simplest example of a PPT operator that is not positive is the alternation operator, defined in case $H=K$ by $A(\phi\otimes\psi)=\psi\otimes\phi$. Notice that if $H=K$ has dimension n, then $(1/n)A$ defines a state on $\mathscr{F}$, namely, $\omega(\phi\psi)=(1/n)|\langle\phi,\psi\rangle|^2$.

The following example shows that a compact PPT operator on $H\otimes K$ may have finite trace even though not trace-class.

Example 3.9. Let $H=\bigoplus_k H_k$, where H_k is k-dimensional. Let A_k be the alternation operator on $H_k\otimes H_k$. Define an operator A on $H\otimes H\simeq \bigoplus_{k,j} H_k\otimes H_j$ by

$$A=\bigoplus_k \frac{1}{k^3} A_k$$

[understanding that $\bigoplus_{k=j} H_k\otimes H_j\subseteq \mathrm{Ke}(A)$]. Then A is a norm limit of finite-rank operators, hence, compact; A is PPT and has finite trace, but is not trace-class: $|A_k|$ is the $k\times k$ identity operator, which has trace k^2, so the trace of

$$|A|=\sqrt{A}\,A^*=\bigoplus_k \frac{1}{k^3}|A_k|$$

is infinite. (Notice that, although not trace-class, A is Hilbert–Schmidt.)

Thus, the class of operators—even, of compact operators—on $H\otimes K$ that represent states on $\mathscr{F}$ is properly larger than the self-adjoint trace-class of $H\otimes K$.

The problem of characterizing explicitly the PPT operators—even in the simplest case of interest, namely, $\mathbf{C}^2\otimes\mathbf{C}^2$—is nontrivial, and remains unsolved. In particular, no characterization is known of the extreme points of $\Omega(\mathscr{F})$. It is not totally obvious that the usual QM pure states, i.e., the one-dimensional projections on $H\otimes K$, represent pure states in Ω.

4. REPRESENTATION OF STATES BY OPERATORS

We now ask when it is possible to represent a given state ω by an operator on $H \otimes K$. It does not follow from Proposition 3.6 that every state is representable by an operator on $H \odot K$, let alone by one on $H \otimes K$. We will obtain a representation instead by operators on $H \otimes \bar{K}$.

Lemma 4.1. To every state of $\mathscr{F}$ there corresponds a unique pair Λ, $\Lambda^{\sim}$ of weak*-to-WOT continuous linear operators

$$\Lambda: \quad \mathscr{B}_h(H) \to \mathscr{B}_{1,h}(H)$$

$$\Lambda^{\sim}: \quad \mathscr{B}_h(K) \to \mathscr{B}_{1,h}(K)$$

such that for unit vectors $\phi \in H$ and $\psi \in K$ with corresponding projections P_ϕ and P_ψ,

$$\omega(\phi\psi) = \mathrm{Tr}(\Lambda(P_\phi)P_\psi) = \mathrm{Tr}(\Lambda^{\sim}(P_\psi)P_\phi)$$

Proof. This is merely a translation of Proposition 2.1 into the context of frame manuals. ■

Proposition 4.2. Let $H = K$. Then to every state on $\mathscr{F}$ there corresponds a unique operator Λ on $H \otimes \bar{K}$ such that

$$\omega(\phi, \psi) = \langle \Lambda\phi \otimes \bar{\phi}, \psi \otimes \bar{\psi} \rangle$$

Proof. Recall that $H \otimes \bar{H}$ is isomorphic to the Hilbert-Schmidt class of H by the map $\phi \otimes \psi \mapsto T_{\phi,\psi}$, where

$$T_{\phi,\psi}\eta = \langle \eta, \psi \rangle \phi$$

Extend Λ_ω to all of $\mathscr{B}(H)$ by the Cartesian decomposition. Then note that Λ_ω is determined by its values on finite-rank operators, hence also by its restriction to the Hilbert-Schmidt class. The range of λ_ω is contained in the trace-class, hence, in the Hilbert-Schmidt class. It is clear from Proposition 4.1 that $\Lambda_\omega^{\sim}$ functions as an adjoint for Λ_ω, so, by the Hellinger-Toeplitz theorem, the latter is bounded. Finally, since the isomorphism $\phi \otimes \psi \mapsto T_{\phi,\psi}$ identifies $\phi \otimes \bar{\phi}$ with the projection P_ϕ, Λ_ω represents ω in the desired way. ■

If $H \neq K$, then with no loss of generality, suppose U is a unitary map from K into H. We then obtain, for each ω, an operator Λ on $H \otimes K$ such that for all unit vectors ϕ and ψ,

$$\omega(\phi\psi) = \langle \Lambda(\phi \otimes U^*\bar{\phi}), U\psi \otimes \psi \rangle$$

We will refer to Λ as the *auxiliary representation* of ω, and write $\Lambda = \Lambda_\omega$. In the case $K \neq H$, this representation depends on an arbitrary choice of the unitary U, but is otherwise unique. To simplify the discussion, we will assume henceforth that $H = K$.

Lemma 4.3. If H is separable, Λ_ω is a compact operator.

Proof. It is known that every bounded linear operator from l_2 to l_1 is compact. Choosing E to be any orthonormal basis for $H \otimes H$ consisting of pure tensors, $l_2(E)$ is isomorphic to the Hilbert-Schmidt class of H and the trace-class of H is isomorphic to a subspace of $l_1(E)$. The Hilbert-Schmidt norm is larger than the operator norm; hence, the l_1-norm of Λ_ω is uniformly bounded on the unit ball U_2 of l_2. Hence, $\Lambda_\omega(U_2)$ is compact in l_1—thus, also compact in l_2. ■

Corollary. Every signed weight on $\mathcal{F} \otimes \mathcal{F}$ is representable by a unique sesquilinear form.

Proof. Let $F(\phi, \psi, \eta, \zeta) = \langle \Lambda_\omega \phi \otimes \bar{\psi}, \eta \otimes \bar{\zeta} \rangle$. This form is bounded in all arguments, linear in ϕ and ψ, and antilinear in η and ζ—hence, lifts uniquely to a sesquilinear form $\Phi(\phi \otimes \psi, \eta \otimes \zeta)$ on $H \odot H$. ■

The state ω is now representable in two ways: by a sesquilinear form Φ on $H \otimes H$ and by an operator Λ_ω and its associated sesquilinear form on $H \otimes \bar{H}$. The two forms may be expanded as forms F and G, respectively, in four variables; then we have

$$\omega(\phi, \psi) = F(\phi, \psi, \phi, \psi) = G(\phi, \bar{\phi}, \psi, \bar{\psi})$$

Evidently, the one form is obtained from the other by permuting the two middle terms and then conjugating in the second and fourth terms. One naturally asks whether the boundedness of $\langle \Lambda \cdot, \cdot \rangle$ can be used to secure that of Φ.

Proposition 4.5. A state ω is representable by a Hilbert-Schmidt operator on $H \otimes K$ iff the auxiliary representation Λ_ω is Hilbert-Schmidt.

Proof. Define a map Ψ: $(H \otimes H) \times (H \otimes H) \to (H \otimes \bar{H}) \otimes (\bar{H} \otimes H)$ by

$$\Psi(\phi \otimes \psi, \eta \otimes \zeta) = (\phi \otimes \bar{\eta}) \otimes (\bar{\zeta} \otimes \psi)$$

To see that such a map exists, note that Ψ arises as the composition

$$\Psi = (1 \otimes A) \circ R^{-1} \circ (1 \otimes A \otimes 1) \circ (1 \times J)$$

where J: $(H \otimes H) \to (\bar{H} \otimes \bar{H})$ is the antilinear conjugation map

$$\eta \otimes \zeta \mapsto \bar{\eta} \otimes \bar{\zeta},$$

R is the operator effecting the reassociation

$$(H \otimes H) \otimes (\bar{H} \otimes \bar{H}) \mapsto H \otimes (H \otimes \bar{H}) \otimes \bar{H}$$

and A is the alternation operator mapping $H \otimes \bar{H}$ onto $\bar{H} \otimes H$. The operators

A and R are bounded, as is the sesquilinear map $\mathbf{1} \times J$. Hence, Ψ is also a bounded sesquilinear map. If Λ_ω is Hilbert-Schmidt, then the sesquilinear form $\langle \Lambda_\omega \cdot, \cdot \rangle$ determines to a bounded linear functional F on $(H \otimes \bar{H}) \otimes (\bar{H} \otimes h)$ via

$$F(\phi \otimes \bar{\psi} \otimes \bar{\eta} \otimes \zeta) = \langle \Lambda_\omega \phi \otimes \bar{\psi}, \eta \otimes \bar{\zeta} \rangle$$

Hence, $F \circ \Psi$ defines a bounded sesquilinear form, and thus, a bounded operator W, on $H \otimes H$, with

$$\langle W\phi \otimes \psi, \phi \otimes \psi \rangle = F(\Psi(\phi \otimes \psi, \phi \otimes \psi)) = (\Lambda_\omega \phi \otimes \bar{\phi}, \psi \otimes \bar{\psi}) = \omega(\phi\psi)$$

i.e., $W \sim \omega$. Since Λ_ω is Hilbert-Schmidt, given any frames E and F for H,

$$\sum_{\phi, \psi \in E, \eta, \zeta \in F} |\langle \Lambda_\omega \phi \otimes \bar{\eta}, \psi \otimes \bar{\zeta} \rangle|^2 < \infty$$

It follows that the sesquilinear form affiliated with $W \sim \omega$ is likewise square-summable over a frame (namely, EF) of $H \otimes H$, and hence, that W is Hilbert-Schmidt. This last argument is symmetric in W and Λ_ω; hence, if ω is known to be represented by a Hilbert-Schmidt operator W, we may conclude that Λ_ω is Hilbert-Schmidt. ■

In particular, the usual QM states, being representable by trace-class operators, have Hilbert-Schmidt auxiliary representatives. It is not out of the question that every PPT operator with finite trace is Hilbert-Schmidt—at any rate, we have no counterexample. The question of whether every state on the tensor of frame manuals is representable by a PPT operator remains open.

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